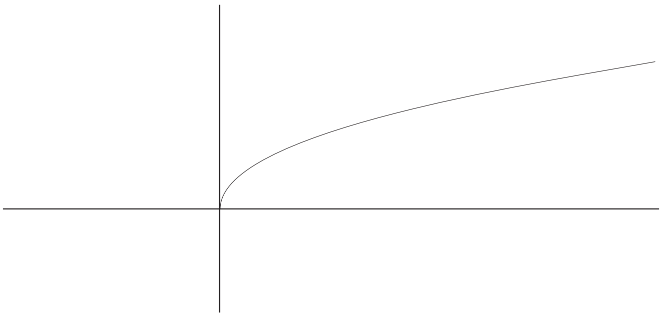
One sided limits.

Can we calculate $\lim_{x\to 0}\sqrt{x}$? We cannot verify the ε - δ definition because any deleted neighbourhood of 0 must be of the form $(-\delta, 0)\cup(0,\delta)$ and so will contain negative x for which $\sqrt{x}$ is not defined. We must conclude, therefore, that $\lim_{x\to 0}\sqrt{x}$ is **not** defined.

Yet, from the graph,



it seems that we could talk meaningfully of the limit of $\sqrt{x}$ as x approaches 0 *from the right.*

Definition 1.1.15 ***Right hand limit****: Suppose that $f : A \to \mathbb{R}$ is defined for x to the right of $a \in \mathbb{R}$, i.e. in an interval $(a, a+\alpha)$ for some $\alpha > 0$. Then*

$$\lim_{x\to a^+} f(x) = L$$

if, and only if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a < x < a+\delta$ then $|f(x) - L| < \varepsilon$. That is:

$$\forall\varepsilon > 0, \exists\,\delta > 0, \forall x \in A, a < x < a+\delta \implies |f(x) - L| < \varepsilon.$$

We sometimes write $f(a^+)$ for $\lim_{x\to a^+} f(x)$.

Definition 1.1.16 ***Left hand limit****: Suppose that $f : A \to \mathbb{R}$ is defined for x to the left of $a \in \mathbb{R}$, i.e. in an interval $(a-\beta, a)$ for some $\beta > 0$. Then*

$$\lim_{x\to a^-} f(x) = L$$

if, and only if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a-\delta < x < a$ then $|f(x) - L| < \varepsilon$. That is:

$$\forall\varepsilon > 0, \exists\,\delta > 0, \forall x \in A, a-\delta < x < a \implies |f(x) - L| < \varepsilon.$$

We sometimes write $f(a^-)$ for $\lim_{x\to a^-} f(x)$.

Note how in both definitions we exclude $x = a$ as we did in the definition of the (two-sided) limit.

Example 1.1.17 *Verify the definition to prove that* $\lim_{x\to 0^+}\sqrt{x}=0$.

Solution Let $\varepsilon>0$ be given, choose $\delta=\varepsilon^2$ and assume x satisfies $0<x<0+\delta$. For such x we have

$$\left|\sqrt{x}-0\right|<\sqrt{\delta}=\sqrt{\varepsilon^2}=\varepsilon.$$

Hence we have verified the definition that $\lim_{x\to 0^+}\sqrt{x}=0$. ■

The following is the first Theorem of the course. The result is useful when a function is defined in different ways to the left and right of the limit point. It is also useful when showing that a limit does not exist.

Theorem 1.1.18 *Let* $f: A\to\mathbb{R}$ *be a function whose domain contains a deleted neighbourhood of* $a\in\mathbb{R}$. *Then*

$$\lim_{x\to a} f(x)=L$$

if, and only if,

$$\lim_{x\to a^+} f(x)=L \quad \textit{and} \quad \lim_{x\to a-} f(x)=L.$$

So a (two-sided) limit exists if and only if both one-sided limits exist and are equal.

Proof ($\Longrightarrow$) Assume $\lim_{x\to a} f(x)=L$. Let $\varepsilon>0$ be given. Then $\lim_{x\to a} f(x)=L$ implies $\exists\,\delta>0:\forall x:0<|x-a|<\delta\Longrightarrow|f(x)-L|<\varepsilon$.

Assume $x:a<x<a+\delta$. Then $0<x-a<\delta$ which implies $0<|x-a|<\delta$. Thus, by the previous line, $|f(x)-L|<\varepsilon$. Hence we have shown that

$$\forall\varepsilon>0,\exists\,\delta>0:\forall x:a<x<a+\delta\Longrightarrow|f(x)-L|<\varepsilon.$$

This is the definition of $\lim_{x\to a+} f(x)=L$.

The verification of $\lim_{x\to a-} f(x)=L$ will follow from the fact that $a-\delta<x<a$ implies $0<|x-a|<\delta$.

($\Longleftarrow$) Assume $\lim_{x\to a^+} f(x)=L$ and $\lim_{x\to a-} f(x)=L$. Let $\varepsilon>0$ be given.

(1) $\lim_{x\to a^+} f(x)=L$ implies $\exists\,\delta_1>0$ such that $\forall x:a<x<a+\delta_1\Longrightarrow |f(x)-L|<\varepsilon$.

(2) $\lim_{x\to a-} f(x)=L$ implies $\exists\,\delta_2>0$ such that $\forall x:a-\delta_2<x<a\Longrightarrow |f(x)-L|<\varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$ and assume x satisfies $0 < |x - a| < \delta$. There are two cases, $x > a$ and $x < a$.

If $x > a$ then, combined with $0 < |x - a| < \delta$, this means $a < x < a + \delta$. Yet $\delta \leq \delta_1$ so $a < x < a + \delta_1$ and thus $|f(x) - L| < \varepsilon$ by (1).

If $x < a$ then, combined with $0 < |x - a| < \delta$, this means $a - \delta < x < a$. Yet $\delta \leq \delta_2$ so $a - \delta_2 < x < a$ and thus $|f(x) - L| < \varepsilon$ by (2).

In both cases $|f(x) - L| < \varepsilon$ and so $0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.

Hence we have verified the definition of $\lim_{x \to a} f(x) = L$. ■

Note you can now see why you need to learn the definitions. In the Theorem you are told that $\lim_{x \to a} f(x) = L$ exists. You can only use this information if you know what saying that the 'limit of f at a is L' means, i.e. what it means *mathematically,* in *symbols*.

Advice for the exams. This is the first of many theorems in this course and it is important that you learn the statements and proofs of them *all*. Perhaps it is too daunting to learn all the proofs so

- start learning them immediately, do not leave revision until the last minute. Unfortunately you won't remember a proof by reading it, you will have to write it out (probably a number of times).
- note that however long a proof it normally contains only one 'idea'. Remember that idea and the rest of the proof often follows.

You can start, though, with learning the *statements* of the proofs. You should attempt to memorise them so well that you can write them down with no thought. As with definitions if I ask for the statement of a Theorem in the exam then that is the opportunity for you to gain easy marks.

The contrapositive of Theorem 1.1.18 is

Corollary 1.1.19 *Let $f : A \to \mathbb{R}$ be a function whose domain contains a deleted neighbourhood of $a \in \mathbb{R}$. Then if either one-sided limit fails to exist at a or they both exist but with different values then the limit at a does not exist.*

For an example of an application of this we start with

Definition 1.1.20 *The **signum** function,* $\operatorname{sgn}(x)$, *is defined for* $x \neq 0$ *by*

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Note that $\operatorname{sgn}(0)$ *is not defined.*

Then, as a second example of a function without a limit at a point we have

Example 1.1.21 *The limit*

$$\lim_{x \to 0} \operatorname{sgn}(x)$$

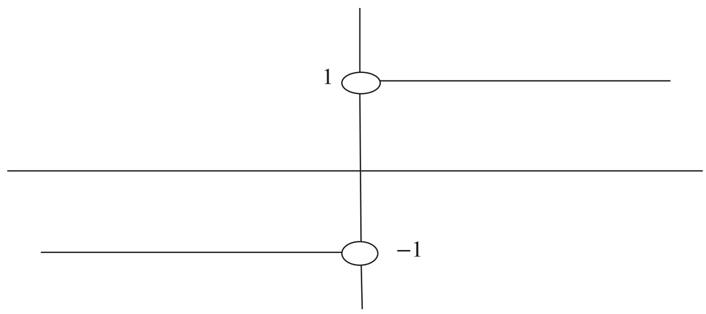
does not exist.

Solution

$$\lim_{x \to 0+} \operatorname{sgn}(x) = \lim_{x \to 0+} 1 = 1 \quad \text{and} \quad \lim_{x \to 0-} \operatorname{sgn}(x) = \lim_{x \to 0-} -1 = -1.$$

Since the limits are different $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist. ■

The graph of $\operatorname{sgn}(x)$ is



Limit of a function at infinity.

Definition 1.1.22 *Assume* $f : A \to \mathbb{R}$ *is defined for all sufficiently large positive* x. *Then*

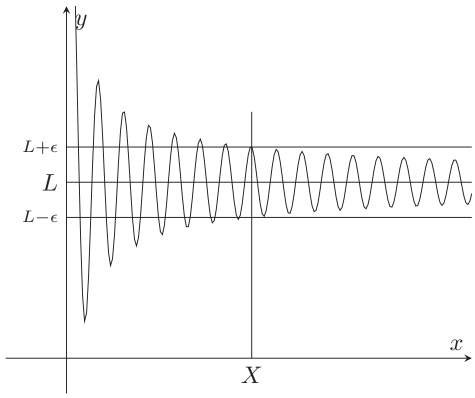
$$\lim_{x \to +\infty} f(x) = L$$

if, and only if, for all $\varepsilon > 0$ *there exists an* $X > 0$ *such that if* $x > X$ *then* $|f(x) - L| < \varepsilon$.

Symbolically

$$\forall \varepsilon > 0, \exists X > 0 : \forall x,\, x > X \implies |f(x) - L| < \varepsilon. \tag{4}$$

A graphical illustration would be



Definition 1.1.23 *Assume $f : A \to \mathbb{R}$ is defined for all sufficiently large negative x. Then*

$$\lim_{x \to -\infty} f(x) = L$$

if, and only if, for all $\varepsilon > 0$ there exists an $X < 0$ such that if $x < X$ then $|f(x) - L| < \varepsilon$.

Symbolically

$$\forall \varepsilon > 0, \exists X < 0 : \forall x,\, x < X \implies |f(x) - L| < \varepsilon.$$

Note 1 The symbols $+\infty$ (also simply known as ∞) and $-\infty$ are not real numbers. You can **not** say

$$\frac{1}{0} = \infty \quad \text{or} \quad \frac{1}{\infty} = 0.$$

The symbols are used in this course simply as a shorthand notation. So $x \to +\infty$ should be read as "x takes arbitrarily large positive values", similarly $x \to -\infty$ is shorthand for "x takes arbitrarily large negative values".

Note 2 In the previous definition we have $X < 0$ and $x < X$. This means that x is negative and is *of greater magnitude than* X, i.e. $|x| > |X|$. So don't think that $x < X$ means that x is "smaller" than X.

Example 1.1.24 *Find*

$$\lim_{x \to +\infty} \frac{x^2}{x^2 + 1},$$

and verify the ε - X definition.

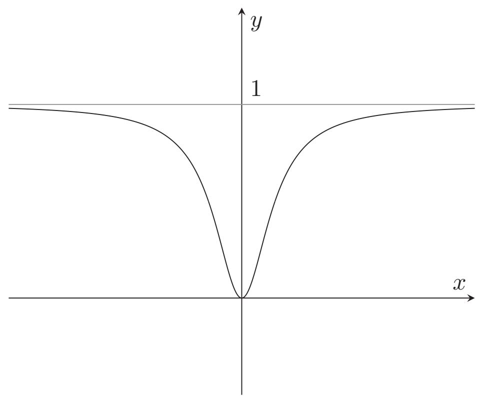
Solution When x is large (in magnitude) then $x^2 + 1$ 'looks like' x^2 and so $x^2/(x^2+1)$ 'looks like' $x^2/x^2 = 1$. So we guess that the limit is 1.

Assume $\varepsilon > 0$ has been given, choose $X = 1/\sqrt{\varepsilon} > 0$ and assume $x > X$. For such x consider

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2}{x^2+1} - 1 \right| \\ &= \frac{1}{x^2+1} < \frac{1}{x^2} < \frac{1}{X^2} \\ &= \frac{1}{(1/\sqrt{\varepsilon})^2} = \varepsilon. \end{aligned}$$

Hence we have verified the ε - X definition that $\lim_{x\to+\infty} x^2/(x^2+1) = 1$. ■

Graphically:



The case of $x \to -\infty$ is left to Tutorial.

Example 1.1.25 *Find*

$$\lim_{x\to+\infty} \frac{x^2}{x^2-1} = 1$$

and verify the ε - X *definition.*

Solution in Tutorial. Assume $\varepsilon > 0$ has been given, choose

$$X = \left(1 + \frac{1}{\varepsilon}\right)^{1/2},$$

and assume $x > X$. For such x consider

$$|f(x) - L| = \left| \frac{x^2}{x^2-1} - 1 \right| = \frac{1}{x^2-1},$$

since $x > 1$. Continuing, since $x > X$ and $X^2 = 1 + 1/\varepsilon$, we have

$$\frac{1}{x^2 - 1} \leq \frac{1}{X^2 - 1} = \frac{1}{(1 + 1/\varepsilon) - 1} = \varepsilon.$$

That is,

$$|f(x) - L| < \varepsilon.$$

Hence we have verified the ε - X definition that $\lim_{x\to-\infty} x^2/(x^2 - 1) = 1$.

■

Note In the proof that $\lim_{x\to-\infty} x^2/(x^2 + 1) = 1$ we made use of the upper bound $1/(x^2 + 1) < 1/x^2$. It would be false to say $1/(x^2 - 1) < 1/x^2$. Instead we could increase the numerator and claim $1/(x^2 - 1) < 2/x^2$ for $x > \sqrt{2}$. This would then lead to the choice of $X = \max\left(\sqrt{2}, 2/\sqrt{\varepsilon}\right)$.

Another example

Example 1.1.26 *Find*

$$\lim_{x\to-\infty} \frac{x}{x+1},$$

and verify the ε - X *definition.*

Solution *Rough work* When x is large (in magnitude) then $x + 1$ 'looks like' x and so $x/(x + 1)$ 'looks like' $x/x = 1$. So we guess that the limit is 1.

Assume $x < X$ with $X < 0$ to be found. In fact, assume $x < -1$ so, in particular, $x + 1 \neq 0$. For such x consider

$$|f(x) - L| = \left|\frac{x}{x+1} - 1\right| = \frac{1}{|x+1|}.$$

We wish this to be $< \varepsilon$. Thus we have to find *upper bounds* on $1/|x + 1|$ or, equivalently, *lower bounds* on $|x + 1|$. For this you could think of the **triangle inequality**, but in the form

$$|a - b| \geq |a| - |b|$$

for all $a, b \in \mathbb{R}$. This gives $|x + 1| \geq |x| - 1$. Thus

$$\left|\frac{x}{x+1} - 1\right| \leq \frac{1}{|x| - 1}.$$

The important observation is that

$$x < X < 0 \Longrightarrow |x| > |X|.$$

This is used in the second inequality in

$$\left|\frac{x}{x+1}-1\right| \leq \frac{1}{|x|-1} \leq \frac{1}{|X|-1}.$$

We now demand that

$$\frac{1}{|X|-1}=\varepsilon, \quad \text{i.e.} \quad |X|=\frac{1}{\varepsilon}+1 \quad \text{or} \quad X=-\left(\frac{1}{\varepsilon}+1\right),$$

since $X<0$.

End of Rough work.

Proof Let $\varepsilon>0$ be given. Choose $X=-(1+1/\varepsilon)$. Assume $x<X$. Then

$$x+1<X+1=-\frac{1}{\varepsilon} \quad \text{so} \quad |x+1|>\frac{1}{\varepsilon} \quad \text{and} \quad \frac{1}{|x+1|}<\varepsilon.$$

So, for such x, we have

$$\left|\frac{x}{x+1}-1\right|=\frac{1}{|x+1|}<\varepsilon.$$

Hence we have verified the ε - X definition that $\lim_{x\to-\infty} x/(x+1)=1$. ■

The proof that $\lim_{x\to+\infty} x/(x+1)=1$ is simpler and left for student.

Note A **common error** will be to use the triangle inequality in the form $|x+1|\leq|x|+1$ and then say

$$\frac{1}{|x+1|} \leq \frac{1}{|x|+1},$$

which is FALSE.

Another **common error** is to say

$$\left|\frac{1}{1+x}\right| \leq \left|\frac{1}{x}\right|.$$

Choose $x=-3$, say, and this says that $1/2<1/3$!

Alternative ending 1 As above

$$|f(x)-L|=\left|\frac{x}{x+1}-1\right|=\frac{1}{|x+1|},$$

but this time demand

$$\frac{1}{|x+1|} \leq \varepsilon.$$

Rearrange as

$$|x+1| \geq \frac{1}{\varepsilon}$$

and open out as either

$$x+1 \geq \frac{1}{\varepsilon} \quad \text{or} \quad x+1 \leq -\frac{1}{\varepsilon}.$$

Since we are looking at large negative x we keep the second inequality

$$x \leq -1 - \frac{1}{\varepsilon},$$

leading to the same X as before.

Alternative ending 2 Again

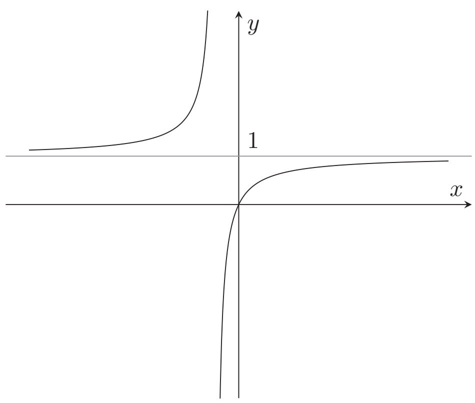
$$|f(x) - L| = \left|\frac{-1}{x+1}\right|.$$

This time we note that for $x < -1$ we have $x+1$ is *negative* and so $-1/(x+1)$ is *positive*. Then

$$\left|\frac{-1}{x+1}\right| = \frac{-1}{x+1}.$$

Demand this is $\leq \varepsilon$, which rearranges again to $x \leq -1 - 1/\varepsilon$.

Graphically $y = x/(x+1)$:



Limits are unique.

Theorem 1.1.27 *Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. If $\lim_{x\to a} f(x)$ exists then it is unique.*

Proof Assume that there exists a function f for which the limit is **not** unique at some point a. Let $\ell_1 < \ell_2$ be two of the different limit values (there may be more than two). In the ε - δ definition of $\lim_{x\to a} f(x)$ choose

$$\varepsilon = \frac{\ell_2 - \ell_1}{3} > 0.$$

Then from definition of $\lim_{x\to a} f(x) = \ell_1$ we find $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies

$$|f(x) - \ell_1| < \varepsilon. \tag{5}$$

Similarly, from the definition of $\lim_{x\to a} f(x) = \ell_2$ we find $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ implies

$$|f(x) - \ell_2| < \varepsilon. \tag{6}$$

Choose $\delta = \min(\delta_1, \delta_2)$ and $x_0 : 0 < |x_0 - a| < \delta$. For such a point both (5) and (6) hold. Hence

$$\begin{aligned} |\ell_2 - \ell_1| &= |\ell_2 - f(x_0) + f(x_0) - \ell_1| \\ &\le |\ell_2 - f(x_0)| + |f(x_0) - \ell_1| \\ &\qquad \text{by the triangle inequality,} \\ &< \varepsilon + \varepsilon \quad \text{by (5) and (6)}, \\ &= 2\varepsilon \\ &= 2\,|\ell_2 - \ell_1|/3. \end{aligned}$$

Dividing through by $|\ell_2 - \ell_1| \neq 0$ we get $1 < 2/3$, a contradiction. Hence the assumption is false and so, if it exists, $\lim_{x\to a} f(x)$ is unique. ■

Note you can now see why you need to learn the definitions. In the Theorem you are told that $\lim_{x\to a} f(x) = L$ exists. You can only use this information if you know what saying that the 'limit of f at a is L' means, i.e. what it means *mathematically*, in *symbols*.

For one-sided limits we have

Theorem 1.1.28 *Suppose that $f : A \to \mathbb{R}$ is defined for x to the right of $a \in \mathbb{R}$. If $\lim_{x\to a+} f(x) = L$ exists then the limit is unique.*

Suppose that $f : A \to \mathbb{R}$ is defined for x to the left of $a \in \mathbb{R}$. If $\lim_{x\to a-} f(x) = L$ exists then the limit is unique.

Proof left to students, no new ideas are required. ■

For limits at infinity we have

Theorem 1.1.29 *If $f : A \to \mathbb{R}$ is defined for all sufficiently large positive x and $\lim_{x\to+\infty} f(x)$ exists then the limit is unique. If $f : A \to \mathbb{R}$ is defined for all sufficiently large negative x and $\lim_{x\to-\infty} f(x)$ exists then the limit is unique.*

Proof Left to students, no new ideas are required. ■